18 pts.

ASTRONOMY 170

17 February 1965 Problem Set # 1

COMPLETED PROBLEM SETS ARE DUE AT LECTURE ON MONDAY, 1 MARCH. IN GENERAL, NO LATE SETS WILL BE ACCEPTED.

1.a) Give a simple derivation (i.e., do not use spherical Bessel functions) showing that theoretical resolving power $\theta \sim \lambda/d$, where d is the diameter of the aperture of the optical instrument.

b) How is the resolving power affected in practice when one observes a planet from the Earth's surface?

c) How is the resolving power affected in practice when one observes the Earth's surface from an artificial Earth satellite? ((3 pts.))

2. Compute the general expression for pressure, p, as a function of altitude, h, in the atmosphere of a planet of radius R in which the temperature varies linearly with height -- i.e., $T = T_o \left[1+a(h/R)\right]$, where T_o is the surface temperature. ((3 pts.))

3. Assume that at the base of the Martian exosphere the number density of particles is 5×10^7 cm^{-3} and the composition is primarily atomic oxygen. Atomic hydrogen is present in abundance $< 10^{-3}$ that of oxygen by number (fractional abundance). For the temperature of the Martian exosphere, consider two cases: (a) T is constant at 500°K; and (b) T is constant at 1200°K. For each case, graph the hydrogen and oxygen densities as functions of altitude. Determine the overall ~~XXXXXXXXXX~~ atmospheric density at the orbits of Phobos and Deimos.

NOTES: (1) Be sure to allow explicitly for the variation of gravity with altitude.

(2) You should plot two separate graphs for the two temperatures. On each graph will be two curves, one each for H and O. ((5 pts.))

4. There are a number of instructive bounds which can be placed on various physical features of planets which are based on the assumptions of hydrostatic equilibrium and of monotonic density decrease with radius.

Consider a non-rotating, spherical planet with the following physical parameters: R, total radius; M, total mass; $\bar{\rho}$, average density; ρ_c, density at the center; p_c, pressure at the center.

a) Referring to the equation of hydrostatic equilibrium, under what obvious condition will p_c be the smallest for planets of given mass and mean radius? Show that if this condition obtains, then

$$p_c \geq \frac{2\pi G}{3} \bar{\rho}^2 R^2 .$$

b) Under what obvious (but unphysical) condition will p_c be the largest? Show that the implied inequality due to this condition can be combined with that in (a) to give

$$\frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} G M^{2/3} \bar{\rho}^{4/3} \leq p_c \leq \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} G M^{2/3} \rho_c^{4/3} .$$

((4 pts))

(5). The general equation of motion of a neutral fluid in a gravitational field is

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p + \vec{g} \ ,$$

where $\vec{V}$ is velocity, p is pressure, ρ is density, t is time, and $\vec{g}$ is acceleration due to gravity. This equation was first obtained by L. Euler in 1755, and is one of the fundamental relations of fluid dynamics.

a) Show that in a special case, Euler's equation will reduce to the equation of hydrostatic equilibrium. What is the special case?

b) If there is no external force, what can we say about the pressure in the fluid if it is in equilibrium?

c) Assume that there is no significant compression of the fluid under the action of an external force. Integrate the vectorial form of the equation of hydrostatic equilibrium (cf. part a) to obtain an expression for the pressure in terms of the altitude. Designate which of your axes is vertically upward.

((3 pts.))

19 pts.

ASTRONOMY 170

24 February 1965 Problem Set 2

COMPLETED PROBLEM SETS ARE DUE AT LECTURE ON WEDNESDAY, 10 MARCH

1. An ideal gas consists of N molecules each of mass m at a temperature T and contained in a volume V.

1½ a) Calculate the mean speed, $\bar{v}$, of the molecules.

1 b) Calculate the root-mean-square speed, v_{rms}, of the molecules.

1 c) Calculate the most probable velocity, $\vec{v}_m$, of the molecules.

1½ d) Calculate the most probable speed, v_m, of the molecules.

1 e) Show that if v_i is any cartesian component of $\vec{v}$, then $\sqrt{\overline{v_i^2}} = (p/\rho)^{1/2}$. This is Newton's value for the speed of sound in a gas.

2 f) Show that the fraction of molecules that have speeds greater than some arbitrary value v is

$$1 - \mathrm{erf}(x) + \frac{2xe^{-x^2}}{\sqrt{\pi}},$$

where $x = v/v_m$.

HINTS: (1) $d^3\vec{v} = 4\pi v^2 dv$

(2) Integrate by parts.

(3) $\int_0^\infty g\,dy = \int_0^a g\,dy + \int_a^\infty g\,dy$

(4) The error function is defined by

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\xi^2}\,d\xi$$

It is normalized such that $\mathrm{erf}(\infty) = 1$.

((8 pts.))

2. Consider a drag-perturbed satellite orbit. Let T be the component of force along the tangent to the orbit and $\mu = G(m' + m)$, where m' and m are the mass of the planet and satellite, respectively.

The following discussion is exerpted from pp. 244-245 of Fundamentals of Celestial Mechanics by J.M.A. Danby, Macmillan, 1962:

"To find the (effect) of T... , it is possible to proceed from first principles. The change in speed is

$$dv = T dt ;$$

differentiating the energy equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

we find

$$2v dv = \mu \frac{da}{a^2},$$

so that

$$da/dt = \frac{2va^2T}{\mu}."$$

"For a satellite moving close to the Earth's surface, the law of resistance is

$g_0 = 1.1\, g_c$

1.1
1.1
11
11
1.21

1.2
1.1

$$mT = -\frac{1}{2} C_D A \rho v^2,$$

where A is the cross-sectional area. C_D is the (dimensionless) aerodynamic drag coefficient, with value of the order of one, and ρ is the atmospheric density... T... is a force per unit mass."

Is the argument above correct ? If it is, expand and clarify it. If it is not, clearly point out why it is not and what the correct argument should be.

((2 pts.))

3. A hypothetical, ideal monatomic gas consists of N atoms of mass m confined to a cubical volume V of side 2s. The center of the cube is at x = y = z = 0.
At t = 0 the velocity distribution function is given by

$$f(\vec{v},\vec{r})\,d^3\vec{v}\,d^3\vec{r} = \frac{1}{V}\left(1+\frac{z}{s}\right)\frac{1}{8\pi v_0^3}e^{-v/v_0}\,d^3\vec{v}\,d^3\vec{r},$$

where $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$ and v_0 is a given magnitude of speed. Notice that this distribution funtion is not a Maxwell-Boltzmann distribution, so that it is hard to produce. However, assume that it has somehow been produced at t = 0. The normalization is such that

$$\iint f\, d^3\vec{v}\, d^3\vec{r} = 1.$$

1½ a) How many molecules are to the right of the plane x = 0 at t = 0 ?

½ b) What is the total kinetic energy of the gas ?

1 c) What will the distribution function be if you wait a long time, provided the walls are perfectly insulating ? This function may be given to within a multiplicative constant. Do not neglect collisions.

HINTS: (1) Note that $d^3\vec{v}$ is the number of points in velocity space between $\vec{v}$ and $\vec{v} + d\vec{v}$. Thus in terms of speed, v, $d^3\vec{v} = 4\pi v^2 dv$.

(2) $\int_0^\infty \Theta^n e^{-\Theta} d\Theta = n!$

((4 pts.))

4. It is convenient, at times, in the study of planetary atmospheres to introduce an upper limit or cut-off to the atmosphere. In parts a and b below do so for the Earth.

ASSUMPTIONS FOR PARTS a and b :
(1) Magnetopause is at ~14 Earth radii.
(2) Base of exosphere is at ~600 kms.
(3) Composition at base of exosphere is primarily O.
(4) Composition at magnetopause is primarily H.

2 a) Re-intgerate the definition of the base of the exosphere, assuming that the atmospheric cut-off occurs at the magetopause. Find an analtic expression for the number density at the base of the exosphere. Include explicitly the variation of gravity with distance.

2 b) Estimate the error which is ordinarily made in the definition of the exosphere by assuming that theatmosphere extends to infinity.

1 c) Under what conditions for some other planet might an atmospheric cut-off be important?

((5 pts.))

ASTRONOMY 170

10 March 1965 Problem Set # 3

COMPLETED PROBLEM SETS ARE DUE AT LECTURE ON MONDAY, 22 MARCH.

1. (4 points) (a) Calculate the equilibrium pressure of xenon in the lunar atmosphere. Assume that

(1) the surface temperature $\simeq 400^{\circ}K$.

(2) thermal escape is the only loss mechanism.

(3) xenon is produced by the bombardment of the lunar surface by cosmic rays, with a cosmic ray flux of 2 cm^{-2} sec^{-1}.

(4) each cosmic ray magically produces one xenon atom, on the average.

(5) the cosmic ray intensity in space is isotropic.

(b) Why does your answer in (a) disagree with the bound given in lecture for the lunar atmosphere? In seeking the explanation of this discrepancy, compare the equilibrium time -- i.e., the time required for the xenon to accumulate to its equilibrium density -- with the age of the solar system.

2. (6 points) Calculate for the Martian atmosphere:

(a) B_j (b) t_2 (c) μ_j.

Assume that

(1) the lower atmosphere can be considered for this problem to be isothermal, and it consists mainly of molecular nitrogen.

(2) $\overline{T}_{surf} \sim 230^{\circ}K$, $\overline{P}_{surf} \sim 1/20$ atm.

(3) from the altitude at which diffusive equilibrium obtains to the base of the exosphere, the atmosphere is isothermal at $\sim 900^{\circ}K$.

(4) the exosphere is isothermal at $\sim 900^{\circ}K$.

(continued next page)

3. (6 points) Consider a planet whose atmosphere and exosphere originally contains 95% molecular hydrogen, 4% water vapor, and 1% silane (SiH_4) by number. Consider the planet and atmosphere otherwise to have properties similar to those of the Earth at the present time -- i.e., same g, same exosphere temperature, same radius, etc. Neglect the photodissociation and ionization of molecular hydrogen and the ionization of silane and water. Consider the photodissociation times of water and silane, and their chemical reaction times, to be short compared to the escape time of molecular hydrogen.

(a) Find the time when there will be an equal number of water and hydrogen molecules.

(b) Find the subsequent time when water will lose the dominant role in the composition which it has gained in part (a) -- i.e., when there will be little water compared with other substances in the atmosphere. You may define "little" in any convenient way.

(c) How soon will the surface be coated with sand and the atmosphere relatively depleted of silane? Indicate the chemical reactions responsible for these events.

(d) At the time found in part (c), what is the composition of the atmosphere? How long will the planet retain an appreciable amount of this atmosphere?

ASTRONOMY 170

22 March 1965 Problem Set #4

COMPLETED PROBLEM SETS ARE DUE AT LECTURE ON WEDNESDAY, 31 MARCH.

1. (6 points) (a) Find α, k_T, and k_S for a perfect gas.

(b) Find the pressure-temperature relation in a convective fluid for each of the following two equations of state:

(i) $$\frac{p}{p_o} = (\text{const.}) \left[(\rho/\rho_o)^{7/3} - (\rho/\rho_o)^{5/3} \right]$$, which is called the Murnaghan-Birch equation of state; and

(ii) $$p = (\text{const.})(\rho^{5/3}),$$ which is the equation of state for a completely degenerate electron gas obeying Fermi-Dirac statistics.

2. (3 points) (a) Compute to first order the effect of the variation of g with altitude on a dry adiabatic lapse rate.

(b) Is the above effect realistically significant at high altitudes? Explain.

3. (6 points) (a) Compute the number density, n(h), in the Earth's atmosphere at altitudes of 5 and 10 km, assuming adiabatic equilibrium.

(b) Derive the adiabatic scale height, defined in terms of the logarithmic derivative of the adiabatic density profile.

(c) Calculate the value of the adiabatic scale height for a representative average temperature of the troposphere -- say, 250^oK.

(d) See how good an approximation to an adiabatic density profile is provided by the expression

$$n(h) = n_o \exp\left\{-h/H_{ad}\right\} \quad ,$$

where n_o is the number density at the Earth's surface, by using this expression to calculate the densities at 5 and 10 km.

(continued next page)

4. (5 points) (a) In class we obtained a differential equation for the variation of ρ with depth in the interior of a convective Earth. Allowing for the variation of g with depth, integrate the above equation under the assumptions:

(i) k_S = const. ; and

(ii) $k_S = (a + bp)^{-1}$.

(b) Knowing that $\rho_o \approx 3$ gm cm^{-3}, what values of k_S and a and b, respectively, are required to give $\bar{\rho} = 5.5$ gm cm^{-3} for the Earth for the two cases?

5. (5 points) Read pages 82 and 83 of Brandt and Hodge. Assume that:

(1) Convective flux = radiative input;
(2) Flux of radiation at the top of the Earth's atmosphere = 2 cal cm^{-2} min^{-1};
(3) About 40% of the radiation incident on the Earth is reflected back into space.

(a) Using mixing length theory, find how super-adiabatic the Earth's atmosphere can be.

(b) Find the V_{rms} of convective motion.

(c) Determine the mean lifetime of a convective bubble.

ASTRONOMY 170

29 March 1965 Hour Examination

Boltzmann's constant $k = 1.38 \times 10^{-16}$ erg $(K^o)^{-1}$.

Newtonian gravitation constant $G = 6.67 \times 10^{-8}$ dyne cm^2 gm^{-2}.

Mass of hydrogen atom $m = 1.67 \times 10^{-24}$ gm.

1. (30%) Diamonds are found in both iron and stone meteorites. In the laboratory, pressures of about 35,000 atmospheres are required to produce diamonds from graphite. Assume that the diamonds in meteorites were formed under high pressures from graphite.

(a) If the diamonds were produced in the interiors of meteorite parent bodies, what was the characteristic dimension of these parent bodies? What objects in the solar system today have such dimensions?

(b) Alternatively, we can assume that the diamonds were produced when the meteorites impacted the Earth. Are the pressures exerted -- even for a brief period of time -- during such an impact sufficient to produce the diamonds?

2. (30%) The hypothetical planet γ Velorum VIII has a mass 100 times that of the Earth and a radius to the top of its clouds 10 times that of the Earth. At some reference level above the clouds of this planet, the atmosphere is composed almost exclusively of molecular hydrogen. The temperature at this level is so low that vibrational and rotational degrees of freedom are not excited. The actual temperature gradient at this level is 2 K^o/km at the moment of observation. The temperature at the top of the clouds is 200^{o}K.

(a) If no condensation occurs near the reference level, is the atmosphere stable against convection?

(b) If very extensive condensation occurs at the reference level, is the atmosphere stable against convection?

(c) Now assume that there is no condensation above the clouds. What is the maximum distance between the top of the clouds and the tropopause?

(continued next page)

3. (40%) According to the planetary cosmogony of Kuiper, the Earth formed as a massive protoplanet of cosmic composition. Most of the initial mass subsequently escaped into space.

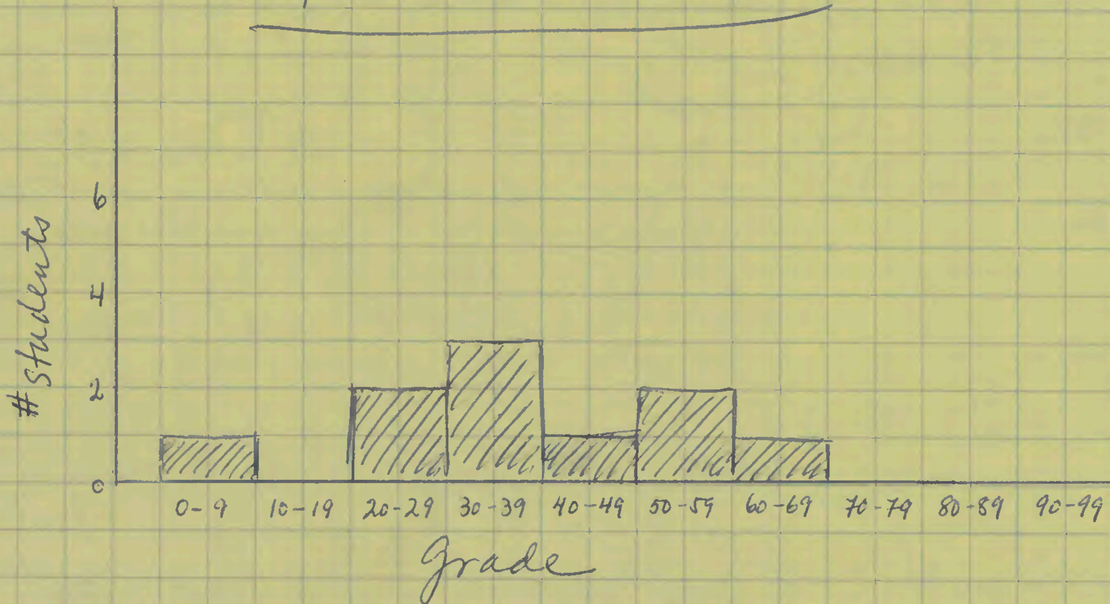
We showed in class that the thermal escape flux from the base of a planetary exosphere, L, can be given by

$$L_j \simeq n_j \left[\frac{g}{2\pi H_j}\right]^{1/2} R e^{-R/H_j} \quad .$$

(a) Show that the above equation implies that because of thermal escape, the functional dependence of the total protoplanetary mass on time follows an exponential decay law. Formulate the problem in such a way that the number density at the base of the exosphere, n_j, does not appear explicitly in the result.

(b) Show that the mass escape rate is a maximum if $R/H \sim 1$. Using the equation derived in (a) for $\mathcal{M}(t)$ and for the case $R/H \sim 1$, compute a rough characteristic time for the Earth to have lost $1 - (1/e)$ of its initial mass. Assume that the mean exosphere temperature $\overline{T}_c \sim 10^3$ °K during this period.

Hour Exam



students
0
2
4
6
0-9
10-19
20-29
30-39
40-49
50-59
60-69
70-79
80-89
90-99
Grade

Astronomy 170

Class Avg. = 39.3

31 March 1965

Soln. Outline to Hour Exam

(1) (a) First must obtain equa. for central pressure, p_c. From hydrostatic equil., $dp/dh = -\rho g \Rightarrow p_c - p_t \simeq p_c \simeq (\bar{\rho})\left(\frac{GM}{R^2}\right)(R) = \bar{\rho}\, GM/R$.

But $\bar{\rho} = M/\frac{4}{3}\pi R^3$ or $p_c \simeq \left(\frac{\bar{\rho}\, G}{R}\right)\left(\frac{4}{3}\pi R^3 \bar{\rho}\right) = \frac{4}{3}\pi \bar{\rho}^2 G R^2$.

$\therefore R \simeq \frac{1}{2\bar{\rho}}\left[\frac{3 p_c}{\pi G}\right]^{1/2}$. $\quad p_c = 3.5 \times 10^4$ atm $= 3.5 \times 10^{10}$ dynes cm^{-2}

$\bar{\rho} \sim 3$ to 5 gm cm^{-3} Let $\bar{\rho} = 3$

Then $R \simeq \frac{1}{2(3)}\left[\frac{3 \times 3.5 \times 10^{10}}{\pi \times 6.67 \times 10^{-8}}\right]^{1/2} = 1.2 \times 10^8$ cm $= 1200$ km.

If $\bar{\rho}$ were larger, R would be smaller. $\boxed{\therefore R \sim 10^3 \text{ km.}}$

This is roughly the size of some of the currently existing planetary moons (eg. our own). Note that Ceres, the largest known asteroid, has a radius ~ 400 km. Thus such an object would be rather sizeable in our solar system.

(b) $F = ma = \frac{d}{dt}(\text{mom.}) \Rightarrow P = \frac{1}{\text{area}}\frac{d(\text{mom.})}{dt}$. $\quad \Delta(\text{mom}) \simeq m v_i \underset{\text{since it stops}}{-} 0$

The initial velocity, v_i, probably would satisfy $V_{esc}^{\oplus} \leq V_i < V_{esc}^{S.S.}$, where $V_{esc}^{\oplus}$ is vel. of escape from the $\oplus$ and $V_{esc}^{S.S.}$ is vel. of escape from the solar system. As a lower bound to the P, let $V_i \sim V_{esc}^{\oplus}$.

If meteorite has radius r and mass m, then area $\simeq \pi r^2$ and $\rho \simeq \frac{m}{\frac{4}{3}\pi r^3}$.

The time from the impact until the meteorite stops completely,

Δt, can be given by $\Delta t \sim nr/V_{esc}^{\oplus}$, where $n \sim 1$ to 10. That is, the meteorite stops after having penetrated some few of its own radii deep. This is roughly what one finds for the pitting on the lunar surface. Thus

$$P \sim \frac{m(V_{esc}^{\oplus})^2}{n\pi r^2} \sim \frac{\frac{4}{3}\pi r^3 \rho (V_{esc}^{\oplus})^2}{n\pi r^2} = \frac{4\rho}{3n}(V_{esc}^{\oplus})^2.$$ Note that if $n \sim 3$, $P \sim \frac{1}{2}\rho(V_{esc}^{\oplus})^2$

This follows since $P \sim$ energy density (e.g. $P = \frac{1}{3}u$ in stellar structure problems). $\therefore$ $P \sim (\frac{1}{2})(3)(1.12\times10^6)^2 \sim 3\times10^{12}$ dynes cm^{-2} $\sim 3\times10^6$ atm.

Thus the pressure is great enough this way.

(Note: $V_{esc}^{\oplus} = \left(\frac{2GM_{\oplus}}{R^{\oplus}}\right)^{1/2}$)

(2) In lecture it was shown that $\left(\frac{dT}{dh}\right)_{ad} = -\frac{g}{\eta c_p}$, where $c_p = \frac{f+2}{2}R(\mu)$.
(Since no rotation or vibration)

$R(\mu) = k/m\mu$, so that $\left(\frac{dT}{dh}\right)_{ad} = -\frac{g}{\eta}\frac{2}{f+2}\frac{m\mu}{k}$.

$$g(\gamma) = g(\oplus)\left[\frac{M(\gamma)}{M(\oplus)}\left(\frac{R(\oplus)}{R(\gamma)}\right)^2\right] = g(\oplus)\left[10^2 \times \frac{1}{10^2}\right] = g(\oplus)$$

$f = 3$, $m = 1.67\times10^{-24}$ gm, $\mu = 2$, $k = 1.38\times10^{-16}$, $g = 9.80\times10^2$ cm sec^{-2}

Thus $\left(\frac{dT}{dh}\right)_{ad} = -\frac{9.5\times10^{-6}}{\eta}$ K°/cm $= -\frac{0.95}{\eta}$ °K/km

(a) No condensation => dry lapse rate => $\eta = 1$ => $\left(\frac{dT}{dh}\right)_{ad} = -0.95$ °K/km

Criterion for stability is $\left|\frac{dT}{dh}\right|_{ad} > \left|\frac{dT}{dh}\right|_{obs}$. Here $.95 < 2$ => not stable against convection.

(b) Since $\left|\frac{dT}{dh}\right|_{ad}$ decreases as $\eta\uparrow$, the wetter the atmosphere, the more unstable it will be.

(c) $T(h) = [illegible] + \left(\frac{dT}{dh}\right)(h)$. Since unstable against convection, the gradient will be the ad. one. ∴ at tropopause let $T = 0 = 200 + (-.95)(h)$ or $h = 200/0.95 = 210$ km. ∴ Tropopause is $\boxed{< 210\text{km}}$ above the cloud tops

(3) (a) $m'L = -\frac{1}{4\pi R}\frac{d\mathcal{M}}{dt}$, where m' is the mass of the dominate element that is leaving

For cosmic composition, $m' \approx$ mass hyd. atom $\equiv m$. Note that $\sigma n H = 1$ & $H = \frac{k}{m}$

$$\frac{d\mathcal{M}}{dt} \cong -4\pi R^2 m L \cong -\frac{4\pi R^2 m}{n}\left[\frac{g}{2\pi H}\right]^{1/2} R e^{-R/H} = -\left(\frac{4\pi R^2 m}{\sigma}\right)\left[\frac{m}{2\pi kT}\right]^{1/2}\left(\frac{G\mathcal{M}}{R^2}\right)\frac{R}{H}e$$

since $g = G\mathcal{M}/R^2$. Thus $\frac{1}{\mathcal{M}}\frac{d\mathcal{M}}{dt} \cong \frac{G}{\sigma}\left(\frac{8\pi m^3}{kT}\right)^{1/2}\frac{R}{H}e^{-R/H}$.

$$\therefore \boxed{\mathcal{M}(t) \approx \mathcal{M}_0 \exp\left\{-\int_0^t \frac{G}{\sigma}\left(\frac{8\pi m^3}{kT}\right)^{1/2}\frac{R}{H}e^{-R/H}\,dt\right\}}$$

(b) (i) Let $y \equiv \frac{1}{\mathcal{M}}\frac{d\mathcal{M}}{dt}$, $x \equiv R/H$, $c \equiv \frac{G}{\sigma}\left(\frac{8\pi m^3}{kT}\right)^{1/2}$. Then $y = cxe^{-x}$.

$y = y_{max}$ when $dy/dx = 0 = c\left[e^{-x} - xe^{-x}\right] \Rightarrow x = 1 \Rightarrow R/H \sim 1$

(ii) $M \; \frac{\mathcal{M}(t)}{\mathcal{M}_0} \approx e^{-t/\tau}$, where $\tau \equiv \frac{\sigma}{G}\left(\frac{kT}{8\pi m^3}\right)^{1/2}$ for max. case of $R/H \sim 1$. When $t = \tau$, $1 - 1/e$ of initial mass will have been lost. Thus

$$t = \tau = \frac{10^{-15}}{6.67\times10^{-8}}\left[\frac{1.38\times10^{-16}\times10^{3}}{8\pi(1.67\times10^{-24})^3}\right]^{1/2} \cong 5\times10^{20}\text{ sec} \approx 1.6\times10^{13}\text{ yr}$$

Thus $t \sim \boxed{1.6\times10^{13}\text{ yrs.}}$ which is far longer than the age of the solar system.

12 April 196. ASTRONOMY 170 Problem Set #5

COMPLETED PROBLEM SETS ARE DUE AT LECTURE ON WEDNESDAY, 21 APRIL.

1. (5 points) Let p and u denote the radiation pressure and radiant energy density, respectively, of radiation incident on a perfectly absorbing surface. Neglect the emitted radiant flux.

(a) In seeking a relationship between p and u, precisely for what volume is "u" to be evaluated? Explain.

(b) Show that for the case of uniform hemispherical radiant flux and total absorption, $p = u/3$.

(c) Derive a relation between p and u for the case that a beam of radiation is normally incident on a perfectly absorbing surface.

2. (4 points) In 1884, Boltzmann deduced theoretically that the total rate of emission of radiant energy by an ideal radiator (blackbody) is proportional to the 4th power of the Kelvin temperature. It can be shown from this that the radiant energy density, u, within an enclosure whose walls are at a uniform temperature, is also proportional to T^4.

Since blackbody radiation can be described by the coordinates p, V, and T, it may be treated as a chemical system, and thermodynamics may be applied to it.

(a) Show from thermodynamic relations derived in class that if U is the internal energy of a chemical system, then

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \quad .$$

The above equation, known as the energy equation, enables us to draw conclusions about U for any system where equation of state is known.

(b) By applying the energy equation to the case of blackbody radiation enclosed in an evacuated container whose walls are perfectly-reflecting thermal insulators, show that

$$u = bT^4 \quad ,$$

where b is a constant. From this expression, derive the blackbody emission flux.

CONTINUED NEXT PAGE

ASTRONOMY 170, Problem Set #5 (continued):

3. (5 points) (a) Show that the saturation vapor pressure over ice, P_s, can be given approximately by

$$P_s = c_1 \, e^{c_2/T} \, .$$

What assumptions are required in the derivation of the above equation? What are the values of c_1 and c_2?

(b) If the clouds of Venus are composed of ice, how much water vapor should we expect above the clouds? Compare your result with the range of spectroscopic results for the water vapor above the clouds of Venus:

$$10^{-2} \text{ g cm}^{-2} \text{ (Dollfus)} , \quad < 10^{-3} \text{ gm cm}^{-2} \text{ (Spinrad)} .$$

(c) If the clouds of Jupiter are composed of ice, how much water vapor should be expect above them? Should it be detectable spectroscopically?

(d) If the clouds of Jupiter are composed of ammonia, how much ammonia vapor should we expect above them?

HINTS: 1. Latent heat of vaporization for water = 2.83×10^{10} ergs gm^{-1}.
2. The triple point of water is at (0°C, 6 mb).
3. Infrared bolometric temperature of the clouds of Venus = 234°K.
4. Infrared bolometric temperature of the clouds of Jupiter $\sim 190^{\circ}$K.
5. Latent heat of vaporization for ammonia = 1.37×10^{10} ergs gm^{-1}

4. (7 points) (a) Consider the saturation adiabatic process of an air parcel lifted past the level at which it becomes saturated. Show that the First Law of Thermodynamics for this process may be written as

$$- L \, dw_s = c_v dT + p \, d\alpha \, ,$$

where L is the latent heat of vaporization, w_s is the mass of water condensed per unit mass of air, and

$$\alpha = \frac{R(\mu)T}{p} \, .$$

(b) Combine the relation in (a) with the equation of hydrostatic equilibrium to obtain an expression for the saturation adiabatic lapse rate, Γ_s.

(c) Compare the relation derived in (b) with the relation that was derived in class. Calculate a rough value of η for the earth's atmosphere.

5 May 1965

ASTRONOMY 170

Problem Set IV

COMPLETED PROBLEM SETS ARE DUE IN ROOM A-103, HARVARD COLLEGE OBSERVATORY, ON FRIDAY, 14 MAY.

1. (6 points) (a) Consider the case of a step driving-function for the surface temperature of a plane parallel, semi-infinite slab, i.e.

$$T = 0 \quad \text{for } 0 \leq t < \frac{P}{2}$$

$$T = T_o \quad \text{for } \frac{P}{2} \leq t < P \; .$$

Solve the one-dimensional equation of heat conduction to obtain the temperature at all depths and times.

(b) Approximate the step driving-function in (a) by the first few terms -- perhaps 4 or 5 -- of a Fourier series. Show graphically how well the step function is approximated. With the series formulation of the step driving-function, solve the one-dimensional heat conduction equation again and compare the result with the result obtained in (a).

2. (4 points) Radioactive potassium, thorium, and uranium in concentrations found in chondritic meteorites release $\sim 10^{-14}$ cal cm^{-3} sec^{-1} of heat. Consider a large, initially "cold" object of chondritic composition in interstellar space. Assume its thermal conductive properties to be those of any rocky material.

Neglecting heating by starlight, compute its equilibrium temperature.

3. (3 points) Consider the Planck distribution for the two special cases: (i) long wavelengths and high temperatures; and (ii) short wavelengths and low temperatures. At surface temperatures of terrestrial planets and at microwave frequencies, which of these approximations is valid? What are the bounds on the wavelength for which this approximation is valid for these temperatures?

5 May 1965

ASTRONOMY 170

Problem Set #8

COMPLETED PROBLEM SETS ARE DUE IN ROOM A-103, HARVARD COLLEGE OBSERVATORY, ON FRIDAY, 14 MAY.

1. (7 points) (a) Murray, Wildey, and Westphal (Astrophys. J. 139: 986, 1964) have obtained brightness temperature maps of Jupiter in the 8-14 micron region. Using the approximation to the Planck distribution, in their wavelength and temperature regimes, of $B_W \propto T^n$, where $n \simeq 8.5$, determine from their Figure 1 an approximate limb-darkening law of the form $I_W \propto \mu^{\alpha}$, where arc cos μ = the angle between the local planetary normal and the line of sight, and α is a constant.

(b) Assume that we are seeing an atmospheric region in the 8-14 micron regime on Jupiter characterized by convective equilibrium and pure absorption. Determine the values of γ, η, and s required to explain the observations and discuss whether the convective-pure-absorption model can plausibly explain the observations.

2. (5 points) (a) Compare the temperature gradient for radiative equilibrium using the Eddington approximation with the temperature gradient for convective equilibrium, under the boundary condition $p \longrightarrow p_o$ as $T \longrightarrow T_o$.

(b) Write down the Schwarzschild instability criterion for this case analogous to the criterion

$$\frac{\eta\gamma(s + 1)}{\gamma - 1} > 4 \quad ,$$

derived in class.

ASTRONOMY 170

21 April 1965 Problem Set #6

COMPLETED PROBLEMS ARE DUE AT LECTURE ON MONDAY, MAY 3rd.

1. (4 points) (a) If the phase law, $\phi(\alpha)$, is symmetrical with respect to the direction of incidence, then

$$q = 2 \int_0^{\pi} \phi(\alpha) \sin\alpha \, d\alpha .$$

Show that this assumption can always be made, even if the actual phase law is asymmetrical, by considering a mean phase law given by

$$\bar{\phi}(\alpha) = \frac{1}{2}\left[\phi(\alpha) + \phi(2\pi - \alpha)\right] .$$

(b) Show that the phase integral q is equal to the volume inside the surface generated by rotation about the polar axis of the phase curve $\phi(\alpha)$ plotted in polar coordinates.

(c) What is the average visual surface brightness (in ergs cm^{-2} sec ...) of Venus when at full phase as seen from the Earth?

HINTS: 1. You may approximate the Cytherean reflection curve by Lambert's law.

2. The surface brightness of a Lambert disk illuminated at normal incidence and at unit distance from the Sun (i.e., 1 A.U.) is, by definition,

$$B = \frac{I_s}{\pi} ,$$

where I_s is the energy flux density received from the Sun at unit distance.

2. (5 points) (a) Fill in the columns for α_{max}, T_{synch}, and $T_{nonsynch}$ in the table for all the planets, the moon, and the asteroids as given in lecture. Use the approximate wavelength-integrated albedos given in class.

(b) Approximately what temperature error would be introduced if we used visual rather than wavelength-integrated albedos for the terrestrial planets?

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3. (3 points) Murray, Wildey, and Westphal (Astrophys. J. 139:986, 1964) have determined a mean brightness temperature of the satellite Jupiter IV in the 8- to 14-μ region as 168.5°K. Compare this brightness temperature with the equilibrium temperature which you expect for Jupiter IV and explain quantitatively why Murray, Wildey, and Westphal found their result to be perplexing.

4. (4 points) (a) Derive the Roche instability criterion for two test particles in synchronous rotation about their primary.

(b) Imagine the Sun surrounded by a nebula of half thickness equal to the radius of Jupiter and which everywhere exceeds the Roche density, so it is gravitationally stable against tidal disruption by the Sun.

What is the mass of the nebula? How does it compare with the total present mass of the planets?